

Multi-level Dynamical Systems: Connecting the Ruelle Response Theory and the Mori-Zwanzig Approach

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Abstract

In this paper we consider the problem of disentangling multi-level systems by connecting the seemingly unrelated approaches of the Mori-Zwanzig projection operator technique and of the Ruelle response theory, for which we propose a new derivation. In a previous paper we have shown that by using the Ruelle response theory on a weakly coupled system it is possible to construct a surrogate dynamics for the slow variables, such that the expectation value of any observable agrees, up to second order in the coupling strength, to its expectation evaluated on the full dynamics, where both slow and fast variables are involved. We show here that such surrogate dynamics agrees up to second order to the effective dynamics one can derive by expanding perturbatively the Mori-Zwanzig projection operator, which creates, instead, an accurate representation of the trajectories of the slow variables. In the case of *e.g.* geophysical fluid dynamics, this implies that the parametrizations of unresolved processes suited for prediction (numerical weather forecast) and those suited for the representation of long term statistical properties (climate) are closely related, if one takes into account, in addition to the widely adopted stochastic forcing, the usually neglected memory effects. This bears relevance for the current trend of aiming at seamless prediction.

1 Introduction

The investigation of multi-level systems is of primary interest for mathematics as well as for natural and social sciences, and is a central task of complexity science. In multi-level systems it is possible to separate the variables into (at least) two subsets, such that the variables with each subset are strongly coupled, while variables belonging to different subsets have a much weaker coupling.

Moreover, in most practical cases, the dynamics of each level takes place in distinct spatial and temporal scales, so that it is hard to define an optimal

resolution for all the variables when we attempt to simulate the system or collect the data: we face a stiff problem [6]. Usually, one is interested in devising ways to account, at least approximately, for the impact of the fast processes occurring at small spatial scales on the slow variables, often describing large scale features, thus defining an effective autonomous dynamics for the slow variables.

If one assumes a vast time-scale separation between the slow variables X and the fast variables Y , the averaging method [3, 9] allows for deriving a dynamics for the X variables. Abramov [1] has recently presented an interested application of this method for deriving a simplified dynamics for a system of geophysical relevance. Furthermore, projector operator techniques have been introduced in statistical mechanics with the goal of effectively removing the Y variables. In particular, a considerable interest has been raised by the Mori-Zwanzig approach, through which a formal - albeit practically inaccessible - solution for the evolution of the X variables (or of any observable $A = A(X)$) is derived [31].

Some of the most outstanding examples of complex systems featuring a multi-level structure can be found in (astro)geophysical fluid dynamics, and, in particular, in the Earth's climate, whose variability spans tens of orders of magnitude in both space and time [12, 26]. As a result, even the most sophisticated climate or weather forecast models cover only a relatively small portion of the range of scales where variability is observed, and, often, one needs to select different approximations and even different formulations of the relevant dynamics depending on the problems under investigation [16, 24, 25]. Moreover, it is common practice and actual necessity to develop parameterizations for taking into account the effects due to the nonlinearity of the equations of the unresolved small scale processes (Y variables) on the scales explicitly resolved by the model (X variables) [15, 27, 8, 7, 3, 24]. Such parametrizations increasingly contain a mixture of stochastic and deterministic formulations [17].

In a previous paper [28] we have approached the problem of defining an accurate effective dynamics for the X variable by focusing on the statistical properties of a general observable $A(X)$ rather than on its trajectory. The starting point has been assuming that the coupling between the X and Y variables is weak, and treating such a coupling as a perturbation to the autonomous dynamics of the X and Y variables, treated as unperturbed system. We have then adopted the response theory developed by Ruelle [21, 23, 10], which allows to compute explicitly how the long-time averages of Axiom A dynamical systems endowed with SRB measure [29] change as a result of small perturbations to the flow [19, 4, 2, 11, 13]. A detailed discussion of the relevance of the Axiom A dynamical systems for the description of actual physical processes is given in [10, 13]. Thus, we have derived explicit formulas to compute the changes in the expectation value of $A(X)$ up to the second order in the (weak) coupling strength between the X and Y variables. Moreover, we have been able to derive a surrogate perturbed dynamics for the system X such up to second order its statistical properties of $A(X)$ are the same as those of the fully coupled (X, Y) system, thus deriving *ab initio* an explicit parametrization of the coupling. The correction due to the coupling entails a deterministic contribution to the dynamics, a stochastic forcing expressed as a sum of multiplicative noise terms with non-trivial correlation properties, and an integral expression which describes a memory term. The unavoidable presence of such memory term suggests that the deterministic and stochastic parametrizations currently used in climate models

have a fundamental flaw which should definitely be addressed.

Geophysics provides further motivations for extending our previous analysis. While climate models aim at capturing the statistical properties of the climate system, weather forecast models try to solve an initial value problem within a finite time horizon. As a side note, one should mention that in more recent times, the prediction problem is cast in a probabilistic rather than deterministic way. Therefore, while both kinds of models face similar problems in terms of impossibility of resolving all relevant scales and require parametrizations, and describe - for the atmospheric part - processes occurring in the same domains, their goals are rather distinct. Nonetheless, recently it has been proposed to assess the quality of climate models by checking their skill for numerical weather prediction. The underlying idea is, loosely speaking, that in order to be able to represent the global properties of the climate attractor, the model should be able to describe well its local properties, which are easier to assess [20]. On a closely related line, currently, a general trend in geophysical fluid dynamics is the pursuit of the so-called seamless prediction [18], whose goal is to develop models able to give convincing forecasts (in a probabilistic sense) for time leads covering very different time scales, ranging from days to years and more, thus covering problems like classical weather prediction to climate projections.

An question left open in [28] was exactly the link between the surrogate dynamics for the X subsystem defined *a posteriori* in order to represent up to second order the coupling between the X and Y variables and the dynamics one would obtain by giving an explicit representation of the Mori-Zwanzig projection operator. As we understand from the discussion above, this, in the specific case of geophysical fluid dynamics, addresses the problem of whether parametrizations developed for weather forecast models can be used for climate models, and whether improvements in weather forecast models can be more or less automatically transferred into climate models. Moreover, as one can easily picture that increasing the resolution of a model amounts to switching on the coupling between the X variables and a larger and larger set of Y variables, what is discussed here has also relevance in the tantalizing effort directed at understanding how the skill of (both climate and weather forecast) models changes with increasing resolution and how parametrizations should be changed when the resolution of a model is increased.

We anticipate that this paper gives the comforting answer that, indeed, the surrogate dynamics derived in [28] and the dynamics projected using the Mori-Zwanzig operator are identical up to second order of perturbation. The key technical tool we use to obtain the proof is a Dyson expansion for describing the evolution of the uncoupled and coupled X and Y systems. While we have no proof for the generic n^{th} order, it is reasonable to conjecture that such agreement exists at all orders.

This article is structured as follows. In Section 2 we present the Dyson expansion for the evolution of the unperturbed and perturbed flows, and provide a way to treat in a unified way the Mori-Zwanzig and Ruelle's approaches. In particular, we present a novel derivation of the Ruelle's response formulas, which provides a simple way to derive the non-perturbative correction for the statistical properties of a general observable, and rewrite the Mori-Zwanzig projection operator in a more treatable form. We also explain how to construct all perturbative terms and define simple rules for constructing diagrams describing the various linear and nonlinear contributions. In Section 3 we then use our new

approach to deal with multi-level systems, thus deriving the explicit expression of the projected dynamics for the X subsystem according to the Mori-Zwanzig formalism up to the second order of perturbation due to the coupling with the Y subsystem. We show that such approximate dynamics agrees with what was obtained in [28] using the Ruelle formalism. In Section 4 we present our conclusions and perspectives for future work.

2 Expanding perturbed flows and averages

Both the Mori-Zwanzig projection operator technique [14, 30] and the response theory of natural invariant measures [21] feature an expansion of evolution operators $\Pi(t) = \exp(Lt)$, where L is a linear differential operator describing the dynamics of observables. As described in *e.g.* [5], this can be easily derived formally in the resolvent formalism, by taking the Laplace transform of such operator exponentials:

$$\mathcal{L}\{\Pi\}(s) = \int_0^\infty dt \exp(Lt) \exp(-ts) = (s - L)^{-1} \quad (1)$$

If L consists of a perturbation around an operator L_0 , i.e. $L = L_0 + L_1$, we can expand the Laplace transform using the equality

$$(A + B)^{-1} = A^{-1} - A^{-1}B(A + B)^{-1}. \quad (2)$$

In the case of the Laplace transform in Eq. 1, we take $A = s - L_0$ and $B = -L_1$, so that the A^{-1} and $(A + B)^{-1}$ terms are themselves Laplace transforms of $\Pi_0(t) = \exp(L_0 t)$ and $\Pi(t)$ respectively. Making use of a non-commutative version of the fact that the Laplace transform of a convolution is the product of the transform, this results in the Dyson expansion of the evolution operator $\Pi(t)$:

$$\Pi(t) = \Pi_0(t) + \int_0^t d\tau \Pi_0(t - \tau) L_1 \Pi(\tau) \quad (3)$$

Another decomposition is possible when making use of the following equality for operator inverses:

$$(A + B)^{-1} = A^{-1} - (A + B)^{-1} B A^{-1}. \quad (4)$$

This gives rise to the following decomposition of $\Pi(t)$:

$$\Pi(t) = \Pi_0(t) + \int_0^t d\tau \Pi(t - \tau) L_1 \Pi_0(\tau) \quad (5)$$

In this section we will use these equations to derive the Mori-Zwanzig and the response theory in a uniform way, so as to highlight their similarities and differences.

2.1 Projection operator techniques

In the case of Mori-Zwanzig a projection is carried out on the level of the observables to remove unwanted, irrelevant and usually fast degrees of freedom. Here

the expansion is performed around the evolution that involves only the relevant part of the phase space. If a dynamical system is defined on a Hilbert space \mathcal{Z} with a relevant subspace \mathcal{X} and its orthogonal complement $\mathcal{Y} = \mathcal{X}^\perp$, then one defines a projection \mathcal{P} of functions on the full phase space to functions on the restricted phase space \mathcal{X} . For example, one can take a conditional expectation with respect to a measure on \mathcal{Z} :

$$(\mathcal{P}A)(x) = \frac{\int_{\mathcal{Y}} A(x, y) \rho(x, y) dy}{\int_{\mathcal{Y}} \rho(x, y) dy}.$$

The derivation given by Zwanzig in Chapter 8 of [31] for a generalized Langevin equation is based on Eq. 5. We write the Liouville equation for an observable A as

$$\frac{dA(t)}{dt} = LA(t) = e^{tL}LA = e^{tL}(\mathcal{P} + \mathcal{Q})LA$$

with $\mathcal{Q} = 1 - \mathcal{P}$. The term involving \mathcal{Q} can be further expanded by making use of Eq. 5 with $L_0 = \mathcal{Q}L$. This gives the following equation

$$\frac{dA(t)}{dt} = e^{tL}\mathcal{P}LA + (e^{t\mathcal{Q}L} + \int_0^t ds e^{(t-s)L}\mathcal{P}Le^{s\mathcal{Q}L})\mathcal{Q}LA$$

In [31] it is argued that this equation is a generalization of the Langevin equation, where the second term is a correlated noise term dependent on the initial conditions of the irrelevant degrees of freedom and the third term represent the memory of the system due to the presence of irrelevant variables that have interacted with the relevant ones in the past. If there is a time scale difference between relevant and irrelevant variables the noise becomes white, while the memory term can in this case be neglected, as the irrelevant variables decorrelate quickly.

2.2 Response theory

We consider as unperturbed system an Axiom A dynamical system endowed with SRB measure [29]. The goal of response theory is different from that of the Mori-Zwanzig technique. Here the averages of observables over a long period of time are of interest:

$$\bar{\rho}(A) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt A(x(t)) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \Pi(t) A(x(0)) \quad (6)$$

Through the evolution of $x(t)$, the evolution operator $\Pi(t)$ again enters into this equation. In the case of response theory the operator L that determines Π consists of a perturbation around an unperturbed evolution: $L = L_0 + L_1$, meaning that we can make use of the expansions 3 and 5 to expand $\Pi(t)$ around $\Pi_0(t)$. The perturbing operator L_1 can derive from an external forcing, or as we will later see from a coupling of internal degrees of freedom. The averages of an observable A for the unperturbed evolution, corresponding to L_0 and Π_0 , will be denoted by $\rho(A)$.

We present here a different derivation than the one presented in [28], where the response formula was derived by iteration Eq. 3 to obtain the different order

response terms. These terms can then be summed over all orders to obtain an expression for the full change in expectation value (as presented in [22]). Here instead we will first derive the equation expressing the full change in expectation value, which can then be expanded in orders of the perturbation.

By inserting Eq. 5 into Eq. 6, we have that

$$\tilde{\rho}(A) = \rho(A) + \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \int_0^t d\tau \Pi(\tau) L_1 \Pi_0(t - \tau) A(x(0)) \quad (7)$$

$$= \rho(A) + \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T d\tau \int_0^{T-\tau} dt \Pi(\tau) L_1 \Pi_0(t) A(x(0)) \quad (8)$$

$$= \rho(A) + \tilde{\rho} \left(\int_0^\infty dt L_1 \Pi_0(t) A \right) \quad (9)$$

or

$$\rho(A) = \tilde{\rho} \left(\left(1 - \int_0^\infty dt L_1 \Pi_0(t) \right) A \right) \quad (10)$$

$$\tilde{\rho}(A) = \rho \left(\left(1 - \int_0^\infty dt L_1 \Pi_0(t) \right)^{-1} A \right) \quad (11)$$

By expanding the resolvent operator, one gets the response terms at different orders of L_1 :

$$\begin{aligned} \tilde{\rho}(A) = \rho(A) &+ \rho \left(\int_0^\infty dt_1 L_1 \Pi_0(t_1) A \right) \\ &+ \rho \left(\int_0^\infty dt_1 \int_0^\infty dt_2 L_1 \Pi_0(t_1) L_1 \Pi_0(t_2) A \right) + \dots \end{aligned} \quad (12)$$

This expression is identical to what is given in [22].

3 Coupled systems

The response theory described in Section 2.2 is quite general, in the sense that we have not defined the form of the perturbing operator L_1 . The perturbation can be a specified external forcing or an internal coupling of degrees of freedom. Here we choose the latter in order to make the comparison to the Mori-Zwanzig formalism. The dynamical system is given by an uncoupled vector field F plus a coupling function Ψ :

$$\begin{aligned} \frac{dX}{dt} &= F_X(X) + \Psi_X(Y) \\ \frac{dY}{dt} &= F_Y(Y) + \Psi_Y(X) \end{aligned} \quad (13)$$

Writing this in terms of observables, we have

$$\begin{aligned} \frac{dA(X, Y)}{dt} &= (L_X(X, Y) + L_Y(X, Y)) A(X, Y) \\ &= (F_X(X) + \Psi_X(Y)) \cdot \nabla_X A(X, Y) \\ &\quad + (F_Y(Y) + \Psi_Y(X)) \cdot \nabla_Y A(X, Y) \end{aligned}$$

where ∇_X and ∇_Y denote the gradients with respect to the variables in X and in Y respectively, $L_X = (F_X + \Psi_X) \nabla_X$ and $L_Y = (F_Y + \Psi_Y) \nabla_Y$.

3.1 Response theory

The response of the unperturbed system to the coupling can be calculated by taking $L_0 = F_X(X) \cdot \nabla_X + F_Y(Y) \cdot \nabla_Y$ and $L_1 = \Psi_X(Y) \cdot \nabla_X + \Psi_Y(X) \cdot \nabla_Y$.

By Eq. 12, the n -th order contribution to the response is calculated by integrating the impact of all n -time couplings over the times between the coupling interactions:

$$\delta^{(n)}\rho(A) = \int d\tau_1 \dots \tau_n \sum_{\substack{i_1, \dots, i_n \\ \in \{X, Y\}}} \delta^{(n)}\rho(A|i_1, \tau_1; \dots; i_n, \tau_n) \quad (14)$$

where

$$\delta^{(n)}\rho(A|i_1, \tau_1; \dots; i_n, \tau_n) = \int \rho_0(dx) L_{1,i_1} \Pi_0(\tau_1) L_{1,i_2} \Pi_0(\tau_2) \dots L_{1,i_n} \Pi_0(\tau_n) A(x)$$

where $x = (X, Y)$ and $L_{1,i}$ represents an interaction affecting the X or Y subsystem, depending on the subscript:

$$\begin{aligned} L_{1,X} &= \Psi_X(Y) \nabla_X \\ L_{1,Y} &= \Psi_Y(X) \nabla_Y \end{aligned} \quad (15)$$

Any infinitesimal contribution to the response can hence be seen as a sequence of couplings that are activated subsequently (i_1 to i_n) and the times between the interactions (τ_1 to τ_n) as depicted in Figure 1.

If one chooses as observable a function A_X that is only dependent on X , all response contributions up to second order are $\delta^{(1)}\rho(A_X|X, \tau)$, $\delta^{(2)}\rho(A_X|Y, \tau_1; X, \tau_2)$ and $\delta^{(2)}\rho(A_X|X, \tau_1; X, \tau_2)$. The first order term $\delta^{(1)}\rho(A|X, \tau)$ is given by

$$\begin{aligned} \delta^{(1)}\rho(A|X, \tau) &= \int \rho_0(dx) L_X \Pi(\tau) A_X(x) \\ &= \rho_{0,Y}(\Psi_X(Y)) \rho_{0,X}(\nabla_X A_X(f^\tau(X))). \end{aligned}$$

The $\delta^{(2)}\rho(A_X|Y, \tau_1; X, \tau_2)$ and $\delta^{(2)}\rho(A_X|X, \tau_1; X, \tau_2)$ terms give

$$\begin{aligned} \delta^{(2)}\rho(A_X|Y, \tau_1; X, \tau_2) &= \rho_{0,Y}(\nabla_Y \Psi_X(f^{\tau_1}(Y))) \rho_{0,X}(\Psi_Y(X) \nabla_X (A_X \circ f^{\tau_2})(f^{\tau_1}(X))) \\ \delta^{(2)}\rho(A_X|X, \tau_1; X, \tau_2) &= \rho_{0,Y}(\Psi_X(Y) \Psi_X(f^{\tau_1}(Y))) \rho_{0,X}(\nabla_X (\nabla_X (A_X \circ f_2^\tau)(f^{\tau_1}(X)))) \end{aligned}$$

Note that since we perturb around the uncoupled system, the unperturbed measure ρ_0 is a product of invariant measures $\rho_{0,X}$ and $\rho_{0,Y}$ on the X and Y subsystems. For this reason and since the operators in 15 are products of a multiplication and derivation operators that commute whenever the dependence is on different variables, each response term can be written as a product of a $\rho_{0,X}$ and $\rho_{0,Y}$ average.

As we have shown in [28], if one collects these first and second order responses to the coupling Ψ , an identical change in expectation values from the unperturbed ρ_0 can be obtained by adding a Y -independent perturbing operator

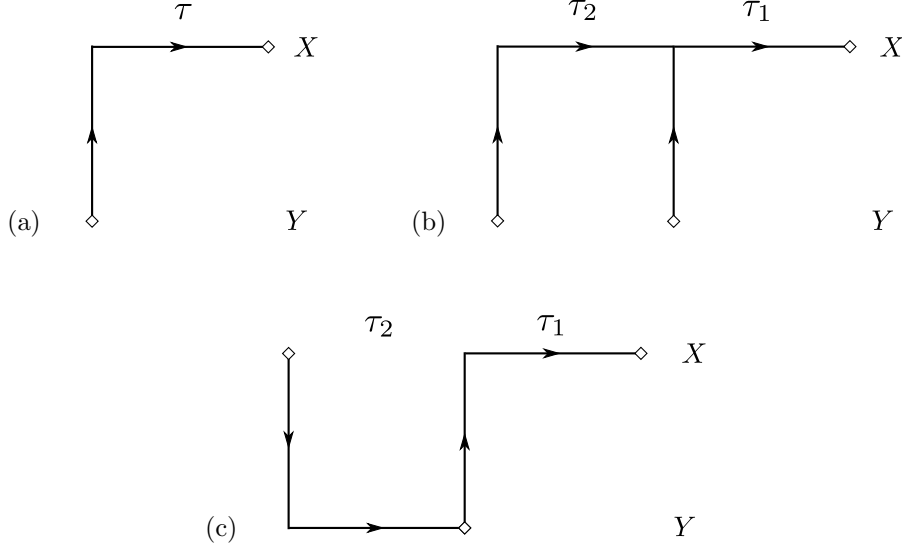


Figure 1: Diagrams representing the response terms $\delta^{(1)}\rho(A_X|X, \tau)$ (diagram (a)), $\delta^{(2)}\rho(A_X|Y, \tau_2; X, \tau_1)$ (diagram (b)) and $\delta^{(2)}\rho(A_X|X, \tau_2; X, \tau_1)$ (diagram (c)).

$L_{1,p}$ to the uncoupled L_0 . It was demonstrated that this can be accomplished by the following dynamical system:

$$\frac{dX(t)}{dt} = F_X(X(t)) + M(X(t)) + \sigma_j(t) + \int_0^\infty d\tau h(\tau, X(t-\tau)) \quad (16)$$

where σ mimics the correlations present in the fluctuations of the coupling from its uncoupled mean:

$$\begin{aligned} \langle \sigma_j(t) \sigma_l(t+\tau) \rangle &= \rho_Y(\Psi'_X(Y) \Psi'_X(f_Y^\tau(Y))), \\ \langle \sigma_j(t) \rangle &= 0 \\ \Psi'_X(Y) &= \Psi_X(Y) - \rho_Y(\Psi_X). \end{aligned} \quad (17)$$

and h is a kernel representing the memory effected by the presence of unresolved variables. Here f_Y^τ denotes the uncoupled evolution of Y generated by F_Y .

The memory term h is given by

$$h(\tau, X) = \Psi_Y(X) \rho_Y(\nabla_Y \Psi_X(f_Y^\tau(Y))). \quad (18)$$

The M term derives from $\delta^{(1)}\rho(A_X|X, \tau)$, the σ term from $\delta^{(2)}\rho(A_X|X, \tau_1; X, \tau_2)$ and the h term from $\delta^{(2)}\rho(A_X|Y, \tau_1; X, \tau_2)$.

It should be noted that the choice of parametrization is not unique. Any time-dependent forcing σ with the correct two-point time-correlations will give the right response up to second order. Also for the memory term there is some freedom. One can also use

$$\frac{dX(t)}{dt} = F_X(X(t)) + M(X(t)) + \sigma_j(t) + \int_0^\infty d\tau h(\tau, f_X^{(t-\tau)}(X_0)) \quad (19)$$

as $X(t-\tau)$ is to zeroth order in Ψ equal to $f_X^{(t-\tau)}(X_0)$. Hence the difference between the two parametrizations will be of order Ψ^3 .

3.2 Direct derivation of surrogate dynamics

We now do a calculation in the style of the Mori-Zwanzig one in Section 2.1 for the dynamical system given in Eq. 13 and for observables A_X that only depend on the relevant variables X . As in Section 2.1, we first do a projection of the evolution equation of A_X to separate X and Y and then expand the evolution of Y .

The evolution equation for A_X is given by

$$\begin{aligned} \left(\frac{d}{dt}A_X\right)(X, Y, t)|_{t=0} &= (L_X A_X)(X, Y) = ((PL_X + QL_X)A_X)(X, Y) \\ &= (F_X(X) + \rho_Y(\Psi_X) + (\Psi_X(Y) - \rho_Y(\Psi_X)))\nabla_X A_X(X) \end{aligned} \quad (20)$$

where $PA(X, Y) = \int_Y \rho_Y(dY)A(X, Y)$. We assume that X and Y start X_0 and Y_0 at time $-t$. We want to find a formal solution for $\Psi_X(Y)$ that we can insert into the previous equation.

The evolution of Ψ_X is given by

$$\begin{aligned} \left(\frac{d}{dt}\Psi_X\right)(X_0, Y_0, t) &= \frac{d}{dt}e^{t(L_X+L_Y)}\Psi_X(X_0, Y_0, 0) \\ &= (L_X + L_Y)\Psi_X(X_0, Y_0, t) \\ &= (F_X(X_0) + \Psi_X(Y_0))\nabla_X \Psi_X(X_0, Y_0, t) \\ &\quad + (F_Y(Y_0) + \Psi_Y(X_0))\nabla_Y \Psi_X(X_0, Y_0, t) \end{aligned}$$

Making use of the decomposition of the Liouvillian $L = L_X + L_Y$ into $L_0(X_0, Y_0) = F_X(X_0)\nabla_X + F_Y(Y_0)\nabla_Y$ and $L_1(X_0, Y_0) = \Psi_X(Y_0)\nabla_X + \Psi_Y(X_0)\nabla_Y$, we get by repeated use of Eq. 3 that

$$\begin{aligned} \Psi_X(Y) &= \Psi_X(X_0, Y_0, t) = e^{tL(X_0, Y_0)}\Psi_X(Y_0) = e^{t(L_0+L_1)}\Psi_X(Y_0) \\ &= e^{tL_0}\Psi_X(Y_0) + \int_0^t d\tau e^{(t-\tau)L_0}L_1 e^{\tau L}\Psi_X(Y_0) \\ &= e^{tL_0}\Psi_X(Y_0) + \int_0^t d\tau e^{(t-\tau)L_0}L_1 e^{\tau L_0}\Psi_X(Y_0) + O(L_1^2) \end{aligned} \quad (21)$$

So we have

$$\left(\frac{d}{dt}A_X\right)(X, X_0, Y_0, t)|_{t=0} = \left(F_X(X) + \rho_Y(\Psi_X) + \tilde{\sigma}(t, Y_0) + \int_0^t d\tau \tilde{h}(\tau, Y_0)\right)\nabla_X A_X(X) \quad (22)$$

where

$$\begin{aligned} \tilde{\sigma}(t, Y_0) &= e^{tF_Y(Y_0)\nabla_Y}\Psi_X(Y_0) - \rho_Y(\Psi_X) \\ \tilde{h}(\tau, X_0, Y_0) &= e^{(t-\tau)L_0(X_0, Y_0)}L_1(X_0, Y_0)e^{\tau L_0(X_0, Y_0)}\Psi_X(Y_0) \end{aligned}$$

Due to the commutation of $F_X\nabla_X$ and $F_Y\nabla_Y$, we have that

$$\begin{aligned} \tilde{h}(\tau, Y_0) &= e^{(t-\tau)(F_X\nabla_X + F_Y\nabla_Y)}(\Psi_X(Y_0)\nabla_X + \Psi_Y(X_0)\nabla_Y)e^{\tau F_Y\nabla_Y}\Psi_X(Y_0) \\ &= \left(e^{(t-\tau)F_X\nabla_X}\Psi_Y(X_0)\right)e^{(t-\tau)F_X\nabla_X}\nabla_Y e^{\tau F_Y\nabla_Y}\Psi_X(Y_0) \end{aligned}$$

If Y_0 starts in the distribution ρ_Y that is invariant under the flow generated by F_Y , then the average of $\tilde{\sigma}$ is zero and the auto-correlation is equal to that of σ given before in Eq. 17:

$$\begin{aligned}\rho_Y(\tilde{\sigma}(t, Y_0)) &= 0 \\ \rho_Y(\tilde{\sigma}(t, Y_0)\tilde{\sigma}(t + \tau, Y_0)) &= \rho_Y(\Psi_X(Y_0)e^{\tau F_Y \nabla_Y} \Psi_X(Y_0))\end{aligned}$$

and

$$\rho_Y(\tilde{h}) = \left(e^{(t-\tau)F_X \nabla_X} \Psi_Y(X) \right) \rho_Y(\nabla_Y e^{\tau F_Y \nabla_Y} \Psi_X(Y_0)) \quad (23)$$

This gives us the memory term of Eq. 19.

4 Summary and Conclusions

The first main result contained in this paper is a new derivation of the Ruelle response theory [21, 23, 10] describing how the statistical properties of Axiom A systems are changed when the underlying dynamics is altered. We have shown that it is possible to obtain directly the exact expression for the expectation value of an observable A computed according to the invariant probability measure of the altered dynamics as the expectation value of another suitably defined observable computed according to the invariant probability measure of the unperturbed dynamics. Such an expression can be expanded at all orders of perturbations, with the general term of order n corresponding exactly to the n^{th} order term obtained by Ruelle through perturbative expansion. Unsurprisingly, the exact result obtained through direct calculation agrees with what was derived by Ruelle through summation of the perturbative series [21].

The Dyson expansion has been instrumental in approaching the problem of studying multi-level systems, by providing a perturbative expansion for the Mori-Zwanzig projection operator [31]. Denoting by X the subset of variables we are interested into and by Y the subset of variables we want to project out, we have derived the effective projected dynamics describing the evolution of an observable of the X variables only up to second order of perturbation. Such a dynamics is identical to the surrogate dynamics for the X variables derived in [28] by imposing, using the Ruelle response theory to describe the impact of the coupling of the X and Y variables, that the expectation value of any observable $A = A(X)$ evaluated on the invariant measure corresponding to the surrogate dynamics agrees up to second order of perturbation to its expectation value evaluated over the complete (X, Y) system. It is important to note that, as discussed in [28], the surrogate dynamics is not unique, because we require agreement only up to second order. This result provides a connection between the Mori-Zwanzig and Ruelle formalisms, which are seemingly different, the first one pertaining to trajectories, the second one to expectation values: we have that if we are able to follow closely (on the average) the individual trajectories, we are also able to model effectively the long-term statistical properties. Such a link between our ability to represent, in some sense, equally well, local and global properties in the phase space strongly relies on the fact that the projection operator leads to introducing a stochastic term and a memory term: the price we have to pay for neglecting the Y degrees of freedom and still retaining a satisfactory representation of the X dynamics on short and long time scales

is going from a deterministic representation in terms of ordinary differential equations to an stochastic representation where integro-differential operators are involved. In particular, the consideration of memory effects marks the difference between what is discussed in this contribution and the classic method of averaging [3, 9], which assumes that there is a vast time-scale separation between the two systems X and Y , so that memory effects are negligible. This is the second relevant result obtained in this paper.

So far, we have been able to prove by direct calculation such a correspondence between the Mori-Zwanzig and Ruelle approaches only up to second order. but it is reasonable to conjecture that the same applies at any order n of perturbation. If this is true, and taking the limit of $n \rightarrow \infty$, one would get that the exact Mori-Zwanzig projected dynamics provides the unique surrogate dynamics for the X variables which is perfectly statistically compatible with the full (X, Y) system for any observables of the X variables only. Therefore, extending the proof we have given to all orders of perturbations would be extremely relevant.

The results presented in this paper have relevance also in the context of the discussion on how to model practically and effectively high dimensional multi-scale systems such as those analyzed by the geophysical fluid dynamical community. In particular, we refer to the problem of a) comparing models featuring different spatial resolutions; and b) constructing so-called parametrizations for the unresolved sub-scale processes. It is clear, from the rather general setting used here that increasing the resolution of a model amounts to enlarging the set of *fast* variables Y (this is particularly evident if one consider Galerkin-like expansions for the fields) coupled to the X *slow* variables, and devising parametrizations is nothing but approximating effectively the surrogated dynamics. This was partly discussed already in [28] in the context of considering exclusively long-term statistical properties (the *climate*). What we have additionally learnt in this paper is that the Mori-Zwanzig projected dynamics, which is instead relevant for reproducing effectively the time evolution (the *weather forecast*) of the slow variables only, provides the surrogate dynamics we need to have a convincing *climate* for the slow variables. This seems to support the idea of assessing the quality of climate models by testing their performance as tools for numerical weather prediction [20], and, more in general, points towards the direction of the so-called seamless prediction [18], which foresees the possibility of using the same models to perform forecasts over very different time scales, ranging from days to years and more. While usually the scholarly literature focuses on stochastic parametrizations as crucial tools in this direction [17], the present work underlines that the consideration of - usually neglected - memory effects is as important for achieving this goal.

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